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COMMENT

On Bäcklund transformations and identities in bilinear form

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Abstract. For bilinear equations of the form $P(D)f \cdot f = 0$ we find all possibilities for rewriting $g^2P(D)f \cdot f - f^2P(D)g \cdot g = 0$ in the form $Q(D)f \cdot g = 0$. This is the first step in finding a Bäcklund transformation.

1. Introduction

In this comment we study Bäcklund transformations in bilinear form. This technique was introduced by Hirota [1, 2]. Let us start with a brief sketch of the method, and formulate the questions that we want to answer. We suppose that we are given an equation in bilinear form

$$P(D)f \cdot f = 0. \tag{1}$$

Here P is a polynomial in, say, n variables, i.e.

$$P(D_1, \ldots, D_n) = \sum c_\alpha D^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha = D_n^{\alpha_n} \cdots D_1^{\alpha_1}$, where

$$D_n^{\alpha_n} \cdots D_1^{\alpha_1} f \cdot g = \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}} \cdots \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} f(x_1 + y_1, \ldots, x_n + y_n) \times g(x_1 - y_1, \ldots, x_n - y_n)\Big|_{y_1 = \cdots = y_n = 0}.$$

To find a Bäcklund transformation, we apply the following trick: consider the equation

$$g^2 P(D)f \cdot f - f^2 P(D)g \cdot g = 0. \tag{2}$$

Then we note that for a solution f, g to (2), the following holds:

f is a solution of (1) $\Leftrightarrow$ g is a solution of (1).

Suppose that we could rewrite (2) in the following form:

$$Q(D)[Q_1(D)(f \cdot g) \cdot Q_2(D)(f \cdot g)] = 0 \tag{3}$$

Then by suitably splitting Q we derive two equations:

$$P_1(D)f \cdot g = 0$$
$$P_2(D)f \cdot g = 0. \tag{4}$$

The system (4) can be called a Bäcklund transformation for $P(D)f \cdot f = 0$. Namely, suppose g is given; then a solution f to equation (4) will also satisfy $P(D)f \cdot f = 0$.

The process of sensibly splitting Q is strongly equation-dependent; it seems unclear in general how to perform it. Two other general questions remain:

1. Is there for any P a solution Q?
2. Is this solution unique? If not, can one find all possibilities?

In this comment we answer these questions. The answer to the first is yes; the proof is already essentially in Hirota [1,2]. This solution is not at all unique. We give all possibilities in terms of a generating identity. The proof that these are all the possibilities is the most difficult part, and not fully included here. For complete proofs, see [3].

2. Algebraic background; partial solution

Let J denote the space of multi-indices $(i_1, \ldots, i_n)$. In practice we will encounter $f^{(i)}$ and $g^{(j)}$ $(i, j \in J)$, which will substitute for the partial derivatives. We will deal with the polynomial algebra $\mathcal{A} = \mathbb{R}[f^{(i)}, g^{(j)}]$, $(i, j \in J)$. In this algebra, we have derivations ∂_r $(r = 1 \cdots n)$ which act in the obvious way. In particular

$$\partial_r(f^{(i)}) = f^{(i+1_r)} \text{ and } \partial_r(g^{(j)}) = g^{(j+1_r)}$$

where $1_r = (0, \ldots, 0, 1, 0, \ldots, 0)$, the rth basis vector.

Corresponding to ∂_r, we introduce the linear map $\mathcal{D}_r$ (the Hirota derivative) as a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, by

$$\mathcal{D}_r(a \otimes b) = \partial_r(a) \otimes b - a \otimes \partial_r(b) \qquad (r = 1, \ldots, n)$$

The derivative ∂_r is also extended to $\mathcal{A} \otimes \mathcal{A}$ by

$$\partial_r(a \otimes b) = \partial_r(a) \otimes b + a \otimes \partial_r(b).$$

Obviously, we can define a Hirota derivative corresponding to any derivation of $\mathcal{A}$.

Lemma 1. Let $\mathcal{D} = \mathcal{D}_r$ and $\partial = \partial_r$ and $\mathcal{A}$ be as above. Then

$$\exp(\epsilon\mathcal{D})(a \otimes b) = \exp(\epsilon\partial)a \otimes \exp(-\epsilon\partial)b.$$

The equality is meant as formal power series, and follows by computing homogeneous terms with reference to ϵ^i, $i = 0, 1, 2, \ldots$.

Note that our definition of $\mathcal{D}_r$ differs from the usual one in the following sense: the image is again in $\mathcal{A} \otimes \mathcal{A}$ and not in $\mathcal{A}$. This is a major difference, as we will see shortly. To get the usual Hirota derivatives D_r we have to project the image of $\mathcal{D}_r$ to $\mathcal{A}$: let $\pi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\pi(\sum a_i \otimes b_i) = \sum a_i b_i$, denote this projection. Then

$$D_r(a \otimes b) := \pi\mathcal{D}_r(a \otimes b)$$

and more generally

$$D_r^k(a \otimes b) := \pi\mathcal{D}_r^k(a \otimes b), \; k = 1, 2, \ldots.$$

The exchange formula is central in this comment.

Theorem 1 (cf Hirota). Let $\partial_\alpha = \sum \alpha_i \partial_i$, $\partial_\beta = \sum \beta_i \partial_i$ and $\partial_\gamma = \sum \gamma_i \partial_i$ and let D_α etc. be the corresponding (projected) Hirota derivatives. Then we have for all $a, b, c, d \in \mathcal{A}$

$$\begin{aligned}\exp(D_\alpha)(\exp(D_\beta)(a \otimes b) \otimes exp(D_\gamma)(c \otimes d)) \\ = \exp(\tfrac{1}{2}(D_\beta - D_\gamma))(\exp(\tfrac{1}{2}(D_\beta + D_\gamma) + D_\alpha)(a \otimes d) \\ \otimes \exp(\tfrac{1}{2}(D_\beta + D_\gamma) - D_\alpha)(c \otimes b))\end{aligned} \tag{5}$$

Proof. The proof is based on lemma 1: expressing all the $\mathcal{D}$ in their corresponding ∂ and projecting on $\mathcal{A}$ gives the result immediately.

Note that this identity solves our first question by taking $a = b = f$ and $c = d = g$. Using the fact that $\exp(\sum \epsilon_i \mathcal{D}_i) = \prod \exp(\epsilon_i \mathcal{D}_i)$, we see that for three multi-indices $k, l, m \in J$, the coefficient of $\alpha^k \beta^l \gamma^m$ in (5) expresses

$$D^k(D^l(f \cdot f) \cdot D^m(g \cdot g))$$

in terms of

$$D^{\cdot\cdot}(D^{\cdot\cdot}(f \cdot g) \cdot D^{\cdot\cdot}(g \cdot f)).$$

Moreover we note that such an expression is not unique. If we take $a = c = f$ and $b = d = g$, then again comparing the coefficients of $\alpha^k \beta^l \gamma^m$ in (5), we see that $D^k(D^l(f \cdot g) \cdot D^m(f \cdot g))$ can be re-expressed. These observations solve a part of our problem; however, they do not solve the most difficult part. It is important to find *all* possibilities, hence all identities of the form

$$D^k(D^l(f \cdot g) \cdot D^m(f \cdot g)) = \sum D^{\cdot\cdot}(D^{\cdot\cdot}(f \cdot g) \cdot D^{\cdot\cdot}(f \cdot g)). \tag{6}$$

This problem will be solved in the next section. The answer is slightly surprising: equation (5) already contains all non-trivial identities!

3. Finding all identities

To study the identities of the form (6) more closely, we introduce two subspaces of $\mathcal{A} \otimes \mathcal{A}$. The first one, denoted by $\mathcal{B}$, is the linear space spanned by the elements $\{f^{(i)} \otimes g^{(j)}\}$, $(i, j \in J)$. Clearly these elements form a basis. Note that $\pi \mid_{\mathcal{B}}$ is injective. This allows us to view $\mathcal{B}$ as a subspace of $\mathcal{A}$. We introduce

$$e_\alpha^{(\beta)} := \partial^\beta \mathcal{D}^\alpha (f \otimes g)$$

which are again elements of $\mathcal{B}$. For these elements one can prove the following lemma.

Lemma 2. $\{e_\alpha^{(\beta)}\}$ is a basis for $\mathcal{B}$.

Proof. The proof (by induction on the number of independent variables n) is based on the observation that $\mathcal{D}^\alpha(f \otimes g) \notin \sum_{r=1}^{n} \mathrm{Im}\,(\partial_r)$.

Identifying $\mathcal{B}$ with a subspace of $\mathcal{A}$, we can define $\mathcal{D}_r$ (and ∂_r) : $\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}\otimes\mathcal{B}$. In $\mathcal{B}\otimes\mathcal{B}$ we define the subspace $\mathcal{C}$, spanned by the elements

$$\mathcal{D}^k(\mathcal{D}^l(f\cdot g)\otimes\mathcal{D}^m(f\cdot g)) \qquad (k,l,m\in J).$$

Thanks to lemma 2, these elements are linearly independent, i.e. they form a basis for $\mathcal{C}$. This shows that finding identities of the form (6) is equivalent to finding $\ker(\pi)\mid_{\mathcal{C}}$. Before turning to $\ker(\pi)$ we mention:

Lemma 3. Let $\mathcal{B}$ and $\mathcal{C}$ be as above. Then $\mathcal{B}\otimes\mathcal{B}=\mathcal{C}\oplus\mathrm{Im}(\partial)$, where $\mathrm{Im}(\partial):=\sum_{r=1}^n\mathrm{Im}(\partial_r)$.

Expressed in normal derivatives, $\ker(\pi)\mid_{\mathcal{B}\otimes\mathcal{B}}$ is easily described.

Lemma 4. $\ker(\pi)\mid_{\mathcal{B}\otimes\mathcal{B}}$ is generated by

1. $f^{(i)}g^{(j)}\otimes f^{(k)}g^{(l)}-f^{(k)}g^{(l)}\otimes f^{(i)}g^{(j)}$
2. $f^{(i)}g^{(j)}\otimes f^{(k)}g^{(l)}-f^{(i)}g^{(l)}\otimes f^{(k)}g^{(j)}$.

The elements mentioned under 1 give rise to trivial identities, namely the identities

$$D^k(D^l(f\cdot g)\otimes D^m(f\cdot g))=(-1)^{|k|}D^k(D^m(f\cdot g)\otimes D^l(f\cdot g)).$$

To analyse the elements under 2, we introduce d_r and $\bar{d}_r$: $\mathcal{B}\to\mathcal{B}$, defined by

$$d_r(f^{(i)}g^{(j)})=f^{(i+1_r)}g^{(j)},\ \bar{d}_r(f^{(i)}g^{(j)})=f^{(i)}g^{(j+1_r)}.$$

The elements $f^{(i)}g^{(j)}\otimes f^{(k)}g^{(l)}-f^{(i)}g^{(l)}\otimes f^{(k)}g^{(j)}$ are precisely the homogeneous terms in

$$\exp\left(\sum_{r=1}^n(\alpha_r d_r+\beta_r\bar{d}_r)\right)f\cdot g\otimes\exp\left(\sum_{r=1}^n(\gamma_r d_r+\delta_r\bar{d}_r)\right)f\cdot g$$
$$-\exp\left(\sum_{r=1}^n(\alpha_r d_r+\delta_r\bar{d}_r)\right)f\cdot g\otimes\exp\left(\sum_{r=1}^n(\gamma_r d_r+\beta_r\bar{d}_r)\right)f\cdot g. \qquad (7)$$

Using $\partial_r=d_r+\bar{d}_r$ and $\mathcal{D}_r=d_r-\bar{d}_r$, we can rewrite (7) in terms of ∂_r and $\mathcal{D}_r$: (7) turns into

$$\exp\left(\frac{1}{4}\sum(\alpha_r+\beta_r+\gamma_r+\delta_r)\partial_r\right)\left\{\exp\left(\frac{1}{4}\sum(\alpha_r+\beta_r-\gamma_r-\delta_r)\mathcal{D}_r\right)\right.$$
$$\left(\exp\left(\frac{1}{2}\sum(\alpha_r-\beta_r)\mathcal{D}_r\right)(f\cdot g)\otimes\exp\left(\frac{1}{2}\sum(\gamma_r-\delta_r)\mathcal{D}_r\right)(f\cdot g)\right)$$
$$-\exp\left(\frac{1}{4}\sum(\alpha_r+\delta_r-\gamma_r-\beta_r)\mathcal{D}_r\right)\left(\exp\left(\frac{1}{2}\sum(\alpha_r-\delta_r)\mathcal{D}_r\right)(f\cdot g)\right.$$
$$\left.\left.\otimes\exp\left(\frac{1}{2}\sum(\gamma_r-\beta_r)\mathcal{D}_r\right)(f\cdot g)\right)\right\}. \qquad (8)$$

Changing variables

$$\begin{aligned} \bar{\delta}_r &= \tfrac{1}{4}(\alpha_r + \beta_r + \gamma_r + \delta_r) \qquad & \bar{\alpha}_r &= \tfrac{1}{4}(\alpha_r + \beta_r - \gamma_r - \delta_r) \\ \bar{\beta}_r &= \tfrac{1}{2}(\alpha_r - \beta_r) & \bar{\gamma}_r &= \tfrac{1}{2}(\gamma_r - \delta_r) \end{aligned} \tag{9}$$

expression (8) transforms into

$$\begin{aligned} \exp\Big(\sum \bar{\delta}_r \partial_r\Big)\Big\{&\exp\Big(\sum \bar{\alpha}_r \mathcal{D}_r\Big)\Big(\exp\Big(\sum \bar{\beta}_r \mathcal{D}_r\Big)(f\cdot g) \otimes \exp\Big(\sum \bar{\gamma}_r \mathcal{D}_r\Big)(f\cdot g)\Big) \\ &- \exp\Big(\frac{1}{2}\sum (\bar{\beta}_r - \bar{\gamma}_r)\mathcal{D}_r\Big)\Big(\exp\Big(\frac{1}{2}\sum (2\bar{\alpha}_r + \bar{\beta}_r + \bar{\gamma}_r)\mathcal{D}_r\Big)(f\cdot g) \\ &\otimes \exp\Big(\frac{1}{2}\sum(-2\bar{\alpha}_r + \bar{\beta}_r + \bar{\gamma}_r)\mathcal{D}_r\Big)(f\cdot g)\Big)\Big\}. \end{aligned} \tag{10}$$

Since (9) is an invertible transformation, the homogeneous terms in (10) span the same space as in (7). For $\ker(\pi)\mid_{\mathcal{C}}$ we only need to consider the coefficients in which $\bar{\delta}$ does not appear (see lemma 3). Hence we are left with the expression between braces, which is identical to theorem 1 for $a = c = f$ and $b = d = g$. So these (and the trivial ones) are all identities.

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